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ACOUSTIC FINITE ELEMENT ANALYSIS OF AXISYMMETRIC FLUID REGIONS

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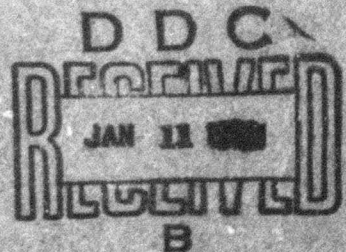Teddington, England

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ABSTRACT

The linear axisymmetric triangular element is used to study the acoustics of axisymmetric fluid regions subject to mixed boundary condtions. Asymmetric acoustic fields are represented by means of Fourier Series expansions in the circumferential coordinate.

22 pp
5 figs

The Finite Element Method (Ref 1) is used to analyse the acoustic field in an axisymmetric fluid region subject to mixed boundary conditions. The simple 'linear' axisymmetric triangular element is used in the analysis, and asymmetric acoustic fields are represented by means of Fourier Series expansions in the circumferential coordinate. Although this note is essentially written from an acoustic point of view, the necessary ingredients are present to facilitate numerical solution of a variety of hydrodynamic potential flow problems, eg, non-axisymmetric flow over bodies of revolution. Section 2 discusses the formulation of the acoustic problem in terms of finite elements and Sections 3 and 4 discuss 'free' vibration and radiation problems. Fortran computer programs, described elsewhere, (Ref 2), were written to solve a variety of problems; some examples are given in Section 5.

2. PROBLEM FORMULATION

Consider the acoustic wave equation in the volume, V, enclosed by the surface, S:

$$\operatorname{div}(\operatorname{grad}\ \phi) + \left(Q - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \right) = 0 \qquad (1)$$

where ϕ is the acoustic velocity potential, related to the excess pressure by the formula,

$$p = \rho\ \frac{\partial\phi}{\partial t}$$

and to the fluid particle velocity by the formula,

$$\underline{v} = -\operatorname{grad}\ \phi$$

C is the sound speed in the fluid and ρ is the density. The term, Q, represents sound sources which may be active in the fluid, eg, a point source of strength, S, say, at the position $\underline{X} = \underline{X}_0$, has the representation,

$$Q = S\ \delta\ (\underline{X} - \underline{X}_0)$$

The boundary conditions are assumed to be of mixed type,

$$\frac{\partial\phi}{\partial n_s} + q + \beta\phi = 0 \qquad (2)$$

where $\dfrac{\partial}{\partial n_s}$ denotes differentiation along the outward normal to the fluid region.

In the usual Finite Element sense, 'stiffness' and 'mass' matrices are

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obtained by minimization of the following functional (Ref. 1):

$$\psi = \frac{1}{2} \int_V \text{grad } \phi \cdot \text{grad } \phi \, dv - \int_V \left(Q - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \right) \phi dv$$
$$+ \int_s \left(q\phi + \frac{1}{2} \beta \phi^2 \right) ds \tag{3}$$

Consider, now, the axisymmetric triangular element of fluid; a cross-section is shown in Fig. 1A, referred to cylindrical coordinates (r, θ, z). Let the variation in ϕ over this element be given by,

$$\phi (r, \theta, z) = \phi_n(r, z) \cos (n\theta) \tag{4}$$

and assume that $\phi_n (r, z)$ has the simple linear expansion

$$\phi_n (r, z) = a_1 + a_2 r + a_3 z \tag{5}$$

It is not difficult to show, in matrix notation, that

$$\phi (r, \theta, z) = [1 \; r \; z] \; [A^{-1}] \begin{bmatrix} \phi_{ni} \\ \phi_{nj} \\ \phi_{nk} \end{bmatrix} \cos(n\theta) \tag{6}$$

and grad $\phi = \left(\dfrac{\partial \phi}{\partial r}, \dfrac{\partial \phi}{\partial z}, \dfrac{1}{r} \dfrac{\partial \phi}{\partial \theta} \right)$

$$= \begin{bmatrix} \cos(n\theta) & 0 & 0 \\ 0 & \cos(n\theta) & 0 \\ 0 & 0 & \sin(n\theta) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dfrac{-n}{r} & -n & \dfrac{-nz}{r} \end{bmatrix} [A^{-1}] \begin{bmatrix} \phi_{ni} \\ \phi_{nj} \\ \phi_{nk} \end{bmatrix} \tag{7}$$

$\phi_{ni}, \phi_{nj}, \phi_{nk}$ are the values of $\phi_n (r,z)$ at the nodal points (i,j,k) of the triangular cross-section, and

$$[A^{-1}] = \frac{1}{\lambda} \begin{bmatrix} r_j z_k - r_k z_j & r_k z_i - r_i z_k & r_i z_j - r_j z_i \\ z_j - z_k & z_k - z_i & z_i - z_j \\ r_k - r_j & r_i - r_k & r_j - r_i \end{bmatrix}$$

$$\lambda = r_j(z_k - z_i) + r_i(z_j - z_k) + r_k(z_i - z_j)$$

Substitution of (6) and (7) into (3) and then minimizing ψ with

respect to ϕ_{ni}, ϕ_{nj} and ϕ_{nk} gives the equations,

$$\left[[M] \frac{\partial^2}{\partial t^2} + [S] + [C] \right] \begin{bmatrix} \phi_{ni} \\ \phi_{nj} \\ \phi_{nk} \end{bmatrix} = \begin{bmatrix} F_{niQ} \\ F_{njQ} \\ F_{nkQ} \end{bmatrix} - \begin{bmatrix} F_{niq} \\ F_{njq} \\ F_{nkq} \end{bmatrix} \tag{8}$$

where,

$$[M] = \frac{\epsilon_n \pi}{c^2} [A^{-1}]^T \int_\Delta \begin{bmatrix} 1 & r & z \\ r & r^2 & rz \\ z & rz & z^2 \end{bmatrix} r\,dr\,dz\, [A^{-1}] \tag{9}$$

$$[S] = \epsilon_n \pi [A^{-1}]^T \int_\Delta \begin{bmatrix} \dfrac{n^2}{r^2} & \dfrac{n^2}{r} & \dfrac{n^2 z}{r^2} \\[2mm] \dfrac{n^2}{r} & 1+n^2 & \dfrac{n^2 z}{r} \\[2mm] \dfrac{n^2 z}{r^2} & \dfrac{n^2 z}{r} & 1+\dfrac{n^2 z^2}{r^2} \end{bmatrix} r\,dr\,dz\,[A^{-1}] \tag{10}$$

$$[C] = \epsilon_n \pi [A^{-1}]^T \int_1 \beta \begin{bmatrix} 1 & r & z \\ r & r^2 & rz \\ z & rz & z^2 \end{bmatrix} r\,dl\,[A^{-1}] \tag{11}$$

$$\begin{bmatrix} F_{niQ} \\ F_{njQ} \\ F_{nkQ} \end{bmatrix} = \epsilon_n \pi [A^{-1}]^T \int_\Delta Q_n \begin{bmatrix} 1 \\ r \\ z \end{bmatrix} r\,dr\,dz \tag{12}$$

$$\begin{bmatrix} F_{niq} \\ F_{njq} \\ F_{nkq} \end{bmatrix} = \epsilon_n \pi [A^{-1}]^T \int_1 q_n \begin{bmatrix} 1 \\ r \\ z \end{bmatrix} r\,dl \tag{13}$$

$$\begin{cases} Q = Q_n(r,z)\cos n\theta \\ q = q_n(r,z)\cos n\theta \end{cases} \tag{14}$$

and β is independent of θ.

The θ-integration (involving $\sin^2 n\theta$ or $\cos^2 n\theta$) has been carried out in (9) through (13); Δ denotes the cross-section in the (r,z) plane and l indicates distance along the boundary line of the cross-section. ϵ_n is equal to 1 for $n \neq 0$, and $\epsilon_n = 2$ when $n = 0$.

The assembly of individual elements to form a region (Fig 1B) reflects the fact that the region functional is the sum of each element functional. The assembly is essentially an organizational problem and is best left to a computer program. A matrix equation of the type.

$$\left[[M] \frac{\partial^2}{\partial t^2} + [S] + [C] \right] \left\{ \phi_{ni} \right\} = \left\{ F_{niQ} \right\} - \left\{ F_{niq} \right\} \qquad (15)$$

results where $\{\phi_{ni}\}$ is a column vector of the values of $\phi_n (r,z)$ at all structure nodes and the matrices $[C]$ and $\{F_{niq}\}$ arise entirely from contributions from the exterior boundary of the region.

A general form of the solution of (1) subject to boundary condition (2) may be stated in the following manner. Let the value of ϕ at a structure node (i) be denoted by ϕ_i, then we have the following expansions :

$$\phi_i = \phi_{oi} + \sum_{n=1}^{\infty} \phi_{ni} \cos(n\theta) + \sum_{n=1}^{\infty} \bar{\phi}_{ni} \sin(n\theta) \qquad (16a)$$

$$Q(r,\theta,z) = Q_o(r,z) + \sum_{n=1}^{\infty} Q_n(r,z)\cos(n\theta) + \sum_{n=1}^{\infty} \bar{Q}_n(r,z)\sin(n\theta) \qquad (16b)$$

$$q(r,\theta,z) = q_o(r,z) + \sum_{n=1}^{\infty} q_n(r,z)\cos(n\theta) + \sum_{n=1}^{\infty} \bar{q}_n(r,z)\sin(n\theta) \qquad (16c)$$

$$\beta(r,\theta,z) = \beta(r,z) \quad \text{independent of } \theta. \qquad (16d)$$

The orthogonality property of the harmonic functions allows reduction from a three-dimensional problem (r,θ,z) to a series of two-dimensional problems (r,z). Assuming that the Fourier Series (16b) and (16c) may be truncated at $n = m$, then $2m + 1$ two-dimensional problems need to be solved separately,

1 involving ϕ_{oi}, Q_o and q_o
m involving ϕ_{ni}, Q_n and q_n ⎫ where i runs from 1 to the
m involving $\bar{\phi}_{ni}$, $\bar{Q}_n$ and $\bar{q}_n$ ⎭ number of nodes in the region cross-section.

The total potential ϕ_i may be found by summation of the series (16a). It should be noted that the [M] and [C] matrices are independent of harmonic number, and the [S] matrix for the $\bar{\phi}_{ni}$ is identical to that of ϕ_{ni}.

Parts of the boundary region where q and β are not specified are automatically set to zero velocity, $\left(\frac{\partial \phi}{\partial n} = 0\right)$, and parts required to have zero pressure ($\phi = 0$) may be constrained by modification of the 'assembled' matrix equation. The integrals involved in (9) through (13) may be computed from analytical expressions or numerical approximations; the problem of the singularity at $r = 0$ may be overcome by giving the relevant nodal r-coordinate a small positive value and forcing the condition, if required, of $\phi = 0$ at this node.

3. PROBLEM OF FREE VIBRATIONS

The problem of free vibrations is solved by setting the RHS of (15) to zero to give the matrix equation, for harmonic number n,

$$\left[-\omega^2 \, [M] + [S] + [C] \right]\left\{\phi_{ni}\right\} = 0 \tag{17}$$

where time variation $\exp(-i\omega t)$ has been assumed.

The relevant boundary condition is, from (2),

$$\frac{\partial \phi}{\partial n_s} + \beta\phi = 0 \tag{18}$$

If the boundary is assumed rigid (or pressure release), then β, and hence $[C]$, vanishes to leave (17) as a standard eigenvalue problem.

If the boundary is locally reacting with acoustic impedance,

$$p/V_n = \rho c[\, R(\omega) - iX(\omega)] \tag{19}$$

where R is the resistance function and X the reactance function, then,

$$\beta = K(X - iR) \,/\, (X^2 + R^2), \quad K = \omega/c \tag{20}$$

and the eigenvalue problem is non-standard. However, at 'low' frequencies it is usually possible to neglect R, and the ratio, K/X, is independent of frequency; hence, we have a standard eigenvalue problem for many cases of practical interest, eg, Helmholtz resonators.

4. PROBLEM OF ACOUSTIC RADIATION

Consider an axisymmetric surface, S, vibrating with known normal velocity (into S), Fig. 2A,

$$V_S = V_n \, (r,z) \cos(n\theta) = - \frac{\partial \phi}{\partial n_s} \tag{21}$$

Surround this surface by a sphere of radius, R, Fig. 2A, such that R is large enough to allow a radiation boundary condition on the sphere surface, viz,

$$\phi \sim \exp(iKR) \cos(n\theta)/R \tag{22}$$

$$\text{on } \frac{\partial \phi}{\partial n_s} = \frac{\partial \phi}{\partial R} = (iK - 1/R) \, \phi \tag{23}$$

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Comparison of the boundary condition (2) with (21) and (23) gives:

$$q = V_n(r,z) \cos(n\theta) \quad \text{on the vibrating surface} \tag{24}$$

$$\beta = (1/R - iK) \quad \text{on the large sphere} \tag{25}$$

The region between the surface, S, and the sphere, R, is divided into triangular elements, Fig. 2B; (24) and (25) substituted into (9) through (14) then give the element matrices for assembling into the region matrix (15), which may be solved by standard techniques to yield the potential amplitude, ϕ_{ni}, at each node. The above analysis is repeated for each harmonic, n, to give the general potential field,

$$\phi_i = \phi_{oi} + \sum_{n=1}^{m} \phi_{ni} \cos(n\theta) + \sum_{n=1}^{m} \overline{\phi}_{ni} \sin(n\theta) \tag{26}$$

caused by boundary vibration,

$$V_S = V_o(r,z) + \sum_{n=1}^{m} V_n(r,z) \cos(n\theta) + \sum_{n=1}^{m} \overline{V}_n (r,z) \sin(n\theta) \tag{27}$$

If active sound sources, Q, are present in the fluid the procedure should be as follows, in the usual acoustic scattering notation.

Let $\phi = [\phi_i(r,z) + \phi_S(r,z)] \cos(n\theta)$ $\tag{28}$

where $\phi_i(r,z)\cos(n\theta)$ is the incident potential associated with the active source distribution, Q, and $\phi_S(r,z)\cos(n\theta)$ is the scattered potential. Assuming that the surface, S, is passive with local impedance $z(S)$, we have the following boundary condition on S,

$$\left(\frac{\partial}{\partial n} + iK/z(S) \right) \left(\phi_i + \phi_S \right) = 0 \tag{29a}$$

or, $\quad \dfrac{\partial \phi_S}{\partial n_s} + \left(\dfrac{\partial \phi_i}{\partial n_s} + iK \phi_i/z(S) \right) + iK \phi_S/z(S) = 0$ $\tag{29b}$

The boundary condition to be applied on R is simply (23),

$$\frac{\partial \phi_S}{\partial n_s} + (1/R - iK) \phi_S = 0 \tag{30}$$

The problem is thus formulated in terms of the scattered potential ϕ_S, with boundary conditions,

$$q = \frac{\partial \phi_i}{\partial n_s} + iK \phi_i/z(S) \left. \begin{array}{c} \\ \\ \end{array} \right\} \quad \text{on S} \tag{31}$$

$$\beta = iK/z(S)$$

$$q = 0$$
$$\beta = (1/R - iK)$$
$$\quad \text{on } R \qquad (32)$$

The source term, Q, is not needed explicitly in the element properties (12), its value is implied in the boundary condition (31).

5. SOME NUMERICAL EXAMPLES

(a) Consider a region of fluid contained in the space between two concentric finite cylinders. A cross-section divided into triangular elements is shown in Fig. 3A; there are 80 elements, the bandwidth of the problem is 13 and the total number of degrees of freedom (nodes) is 55. The boundaries of the region are assumed rigid. The natural frequencies of the region are given by values of W_{rmn} where

$$\phi = [A J_n(Kr) + BY_n(Kr)] \cos\left(\frac{m\pi z}{l}\right) \cos(n\theta) \qquad (33)$$

and $\dfrac{\partial\phi}{\partial r}$ is to vanish at $r=a$, and at $r=b$.

$$K = \sqrt{\left(\frac{W^2_{rmn}}{c^2} - \frac{m^2\pi^2}{l^2}\right)}$$

n identifies the circumferential variation of ϕ
m identifies the axial variation of ϕ
S will identify the radial variation of ϕ

a = 8 cms		b = 10 l = 5		C = 150000	
n	m	S	Finite Element Hz	Analytical Hz	%
0	1	0	15052	15000	0.3
0	2	0	30416	30000	1.4
0	0	1	38387	37572	2.2
0	1	1	41753	40456	3.2
0	3	0	46420	45000	3.2
0	2	1	50720	48080	5.5
1	0	0	2658	2658	0.0
1	1	0	15285	15234	0.3
1	2	0	30532	30118	1.4
1	0	1	38481	37667	2.2
1	1	1	41843	40544	3.2
1	3	0	46495	45078	3.1
1	2	1	50791	48154	5.5
2	0	0	5315	5315	0.0
2	1	0	15963	15914	0.3
2	2	0	30877	30467	1.3
2	0	1	38761	37964	2.1
2	1	1	42099	40811	3.2
2	3	0	46723	45313	3.1
2	2	1	51004	48379	5.8

The above Table shows good agreement between the Finite Element natural

frequencies and those computed from the analytical expression, (33); good agreement was also obtained for the mode shapes. The fall off in accuracy as m and S increase reflects the finite division of the fluid region. Natural frequencies and mode shapes were also obtained for the above geometry with pressure - release boundary, and for rigid and pressure-release boundaries of a finite tube of fluid; again good agreement was obtained between Finite Element and Analytical results.

(b) Consider the Helmholtz resonator shown in Fig. 3B, with boundary assumed rigid except where the opening communicates with the external fluid. The boundary condition at the opening is that of (20) with $R << X$ and set to zero,

$$\beta = K/X \tag{34}$$

and it is assumed that this ratio is independent of frequency in the frequency range of interest. The Helmholtz resonance corresponds to the first natural frequency of the mode $n = 0$. The Finite Element natural frequency, ascertained in cases (i) and (ii) below, is to be compared with 122 Hz measured, 121 Hz computed from an 'improved formula' and 133 Hz from the classical formula which is seriously in error for this shape of resonator, (Ref 3) :

(i) If we choose X ($= 8bK/3\pi$) as the well known formula for the reactance function of a piston vibrating in an infinite baffle the Finite Element computation gives 117 Hz.

(ii) If, from a Finite Element Radiation computer program, we compute the reactance function, X, of the geometry of Fig. 3B vibrating with constant velocity at the opening and zero velocity elsewhere ($X = 1.34K$), we obtain a Helmholtz resonance of 125 Hz.

Bearing in mind that the Finite Element method gives an upper bound on the exact natural frequency, good agreement obtains between measured and computed frequencies.

(c) The numerical computation of acoustic radiation is complicated by the necessity of choosing the radius, R, of the large sphere. If the chosen radius is too small, then incorrect results will be obtained due to the enforcement of a radiation boundary condition where such a condition does not apply; if the radius is too large then the computational effort needed for solution may be excessive. In addition, detailed attention must be given to the results, eg, it may be necessary to refine the triangular subdivision where rapid changes in acoustic pressure occur, and in many cases it is advisable to plot acoustic pressures at centroidal rather than nodal points.

Consider the geometry of Fig. 2A, a cylinder of length 20 cm and radius 5 cm with stationary hemispherical end-caps. The region between the surface and a sphere of radius 38 cm was divided into 392 elements involving 225 nodal points and a bandwidth of 33; on the surface 15 nodal points were chosen. Fig 4 shows nodal acoustic pressures obtained when the cylinder surface pulsates (n=0) at 10,000 Hz with velocity amplitude 0.001 cm/sec into a fluid of density 1.0 and sound velocity 150,000 cm/sec. Fig. 5 shows acoustic pressures at $\theta = 0$ when the cylinder rigidly oscillates

(n=1). For comparison, results obtained by means of fitting spheroidal wavefunctions (Ref 4) are given. Important acoustic parameters are the radiation resistance function, R, and the reactance function, X, defined as follows,

$$\text{Real } \frac{1}{2} \int_S pV^* ds = \frac{1}{2} R\rho c \int_S VV^* ds \tag{35}$$

$$\text{Imag. } \frac{1}{2} \int_S pV \, ds = \frac{1}{2} X\rho c \int_S VV^* ds$$

where * denotes complex conjugate. The first of (35) is the acoustic power radiated and the second is the 'wattless' power.

Harmonic	Finite Element		Spheroidal	Wavefunctions
Number	R	X	R	X
0	0.911	0.284	0.939	0.274
1	0.951	0.350	0.987	0.352

Figs. 4 and 5 and the above table show reasonable agreement between the Finite Element method and the series expansion method using spheroidal wavefunctions.

6. SUMMARY

The Finite Element method has been applied to the acoustics of axisymmetric fluid regions subject to mixed boundary conditions; asymmetric fields being represented by means of Fourier Series expansions. Numerical results suggest that good accuracy may be obtained for interior problems with little judgement from the user of the computer programs. Acoustic/potential flow problems involving circular ducts of varying cross-section and longitudinal impedance can easily be solved by the method presented. The exterior (radiation) problem, however, requires detailed study in order to obtain correct sound levels, especially as frequency increases. It may be possible to overcome this problem by using curved elements to represent boundaries and simple wavefunctions to describe the potential field of an element. However, radiation problems intractable by conventional standards may be solved with good accuracy if close attention is paid to the triangularization of the region concerned, eg radiation from the opening of a circular duct and radiation from a finite ring of arbitrary cross-section.

7. ACKNOWLEDGEMENT

Thanks are due to E J Clement for providing much of the effort needed to write the Fortran computer programs. Computer program specifications may be found in Ref. 2.

J H JAMES (SSO)

ARL/BJR

REFERENCES

(1)   ZIENKIEWICZ, The Finite Element Method, Mc-Graw Hill, 1971.

(2)   CLEMENT E J, Acoustic Analysis of Axisymmetric Fluid Regions :
Finite Element Fortran Program Specifications,
ARL Group R Technical Memorandum No 3, April 1973.

(3)   ALSTER, M, Improved Calculation of Resonant Frequencies of
Helmholtz Resonators, J Sound Vib, Vol 24, No 1, 1972, pages 63-85.

(4)   MURRAY M T, Calculation of Acoustic Radiation Using Spheroidal
Wavefunctions, ARL/M/R26, June 1971.

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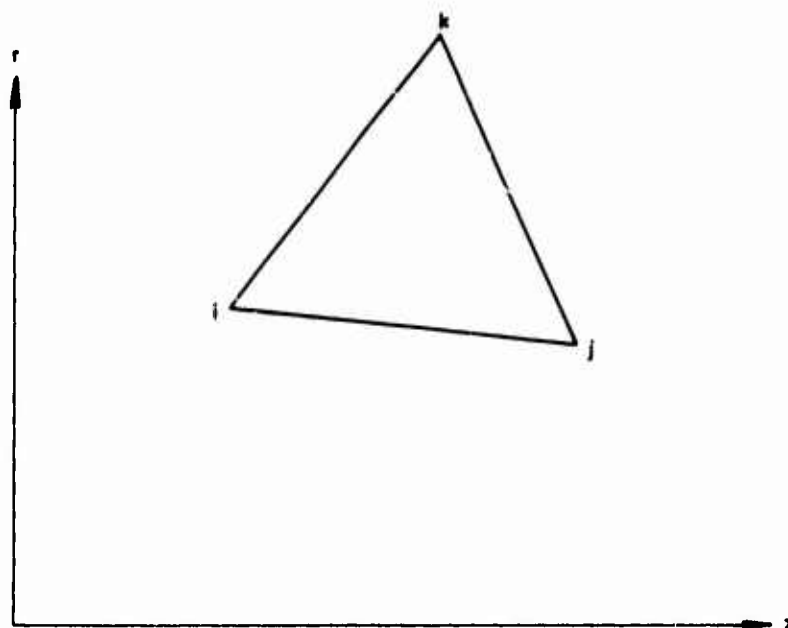
FIG. IA CROSS SECTION OF AXISYMMETRIC TRIANGULAR ELEMENT
REFERRED TO CYLINDRICAL COORDINATES (r, θ, z)
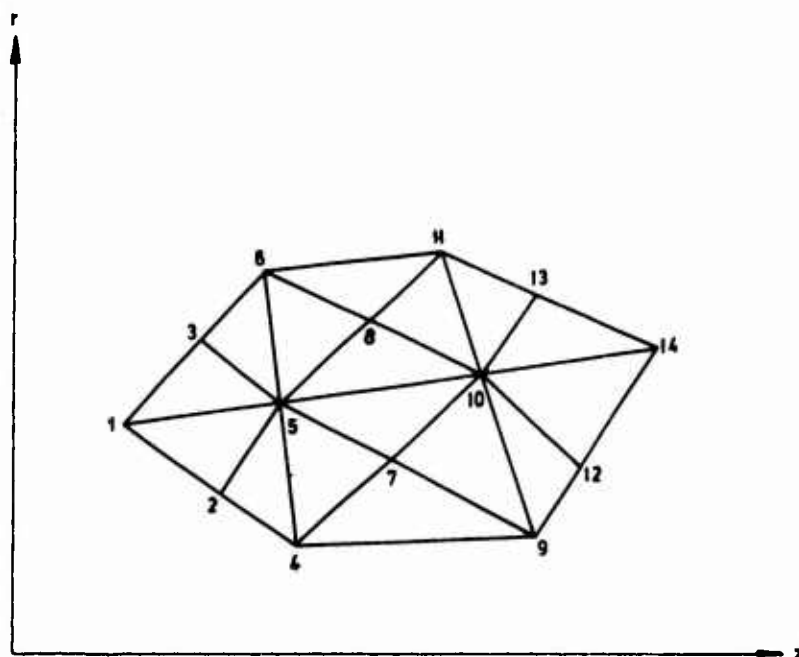
FIG. IB   CROSS SECTION OF REGION DIVIDED
INTO TRIANGULAR ELEMENTS

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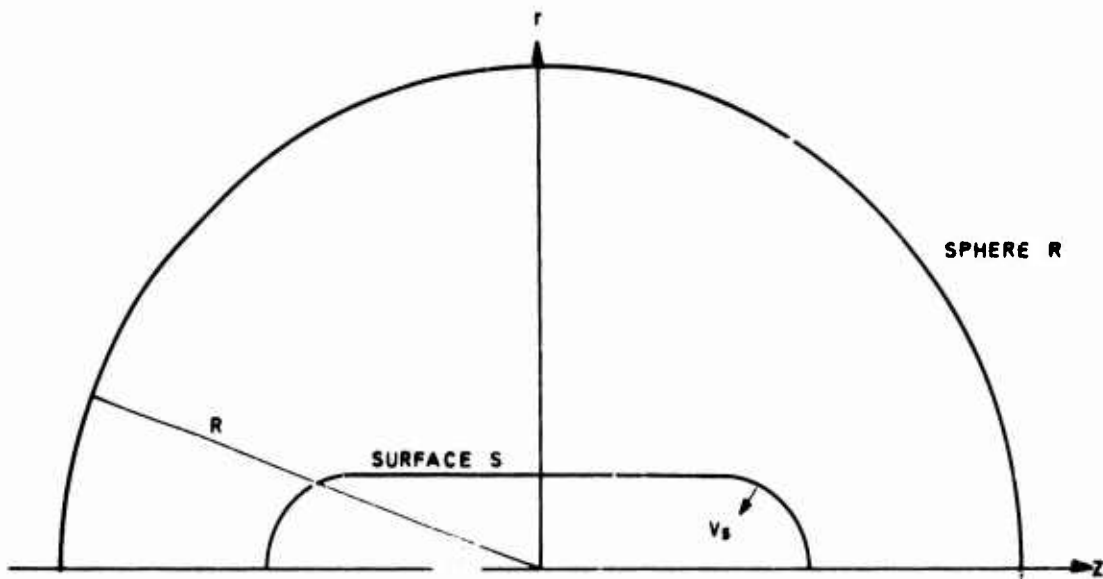
FIG. 2A   CROSS SECTION OF VIBRATING SURFACE SURROUNDED
BY SPHERE OF RADIUS R
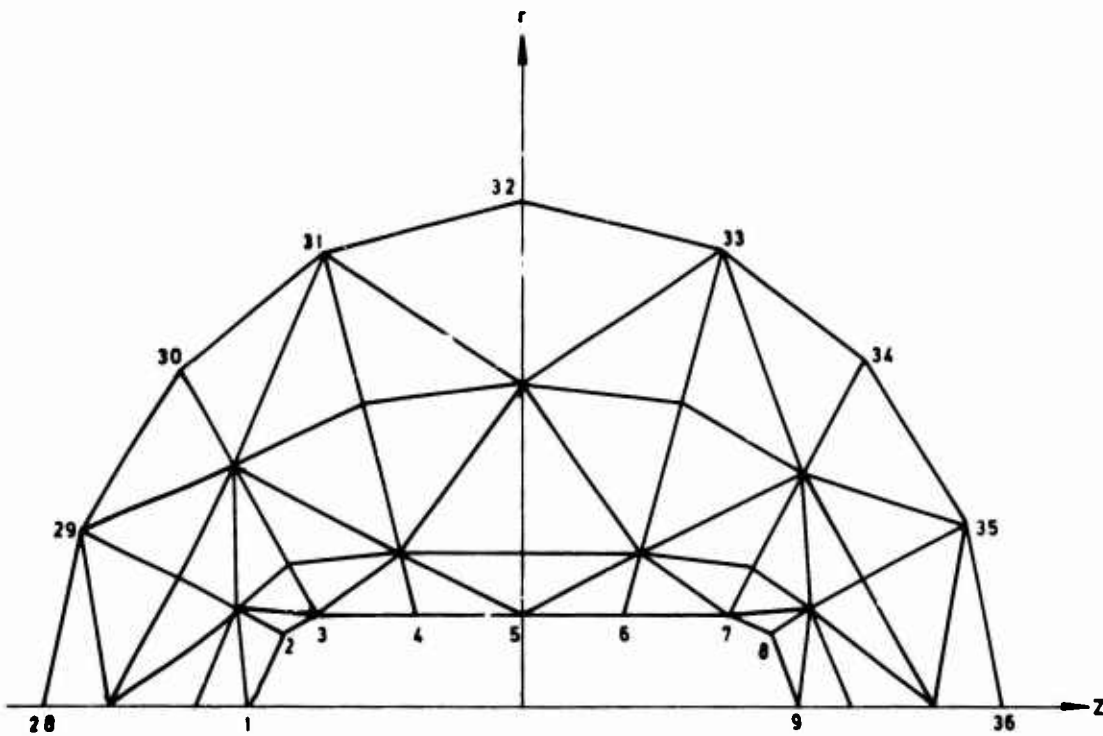
FIG. 2B THE REGION OF FIG. 2A DIVIDED
INTO TRIANGULAR ELEMENTS
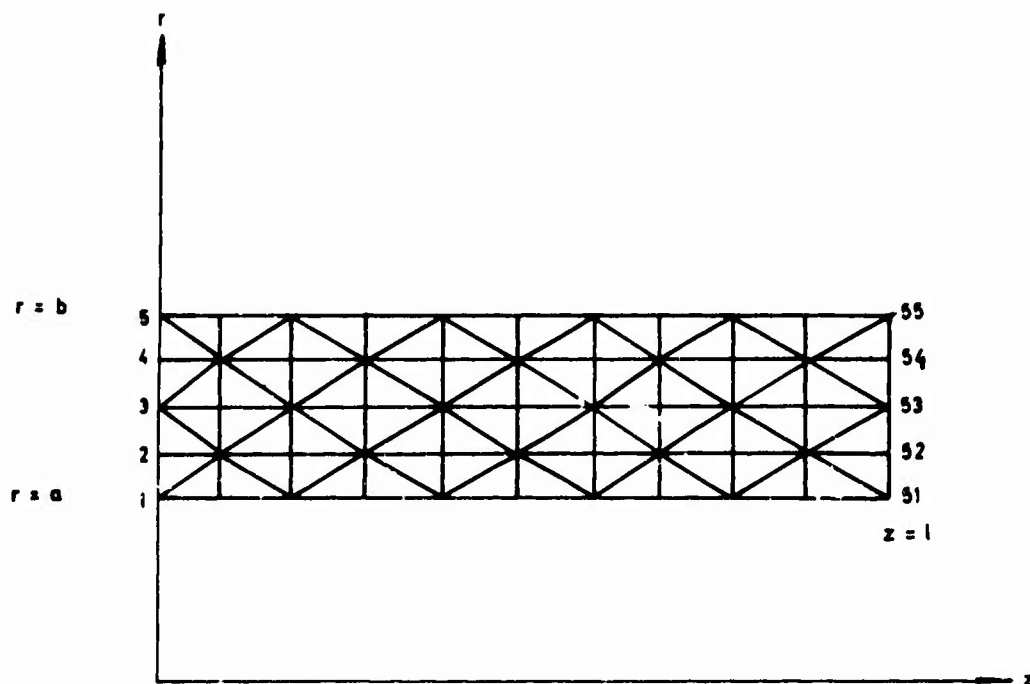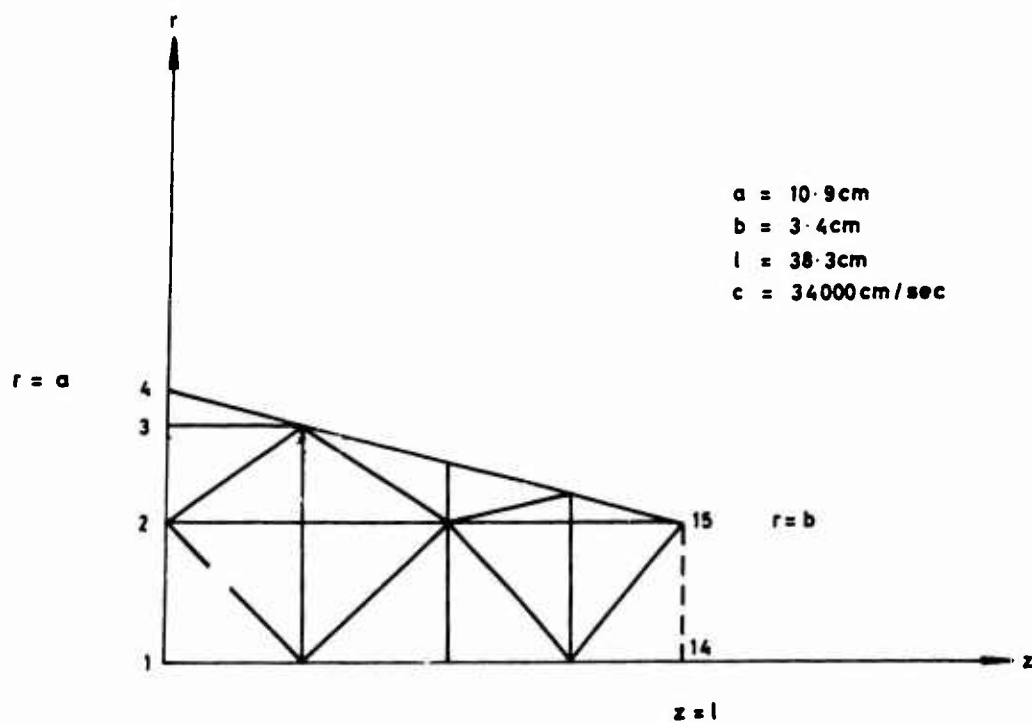
FIG. 3A   CROSS SECTION OF ANNULUS OF FLUID

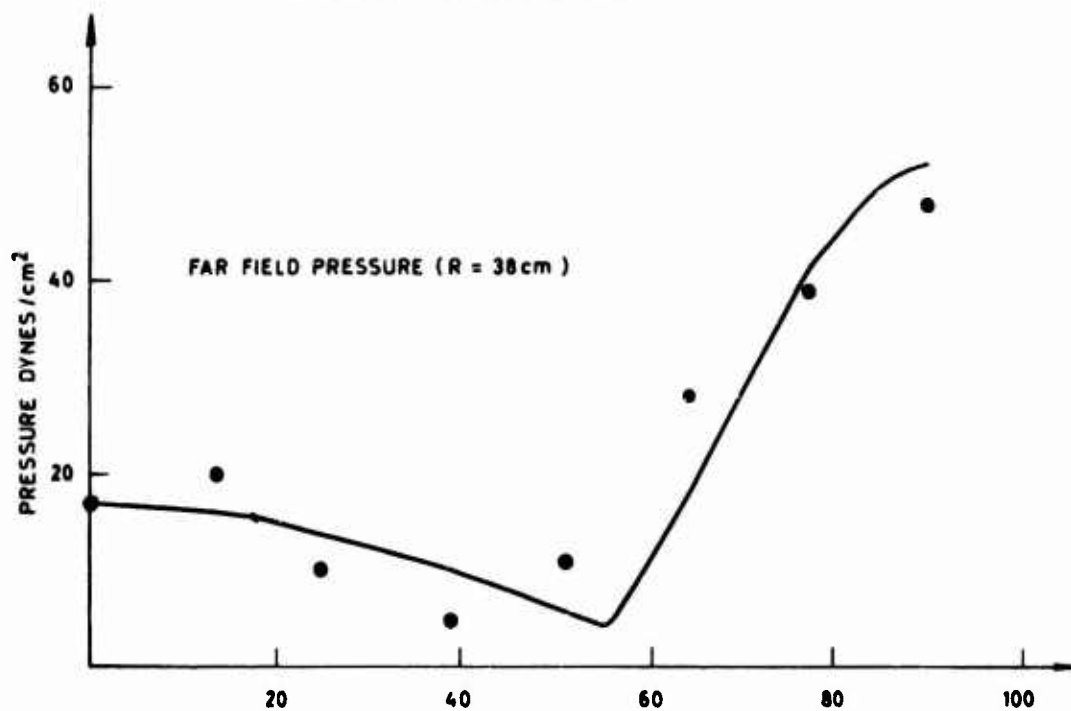FIG. 3B   CROSS SECTION OF HELMHOLTZ RESONATOR
WITH OPENING AT z = l

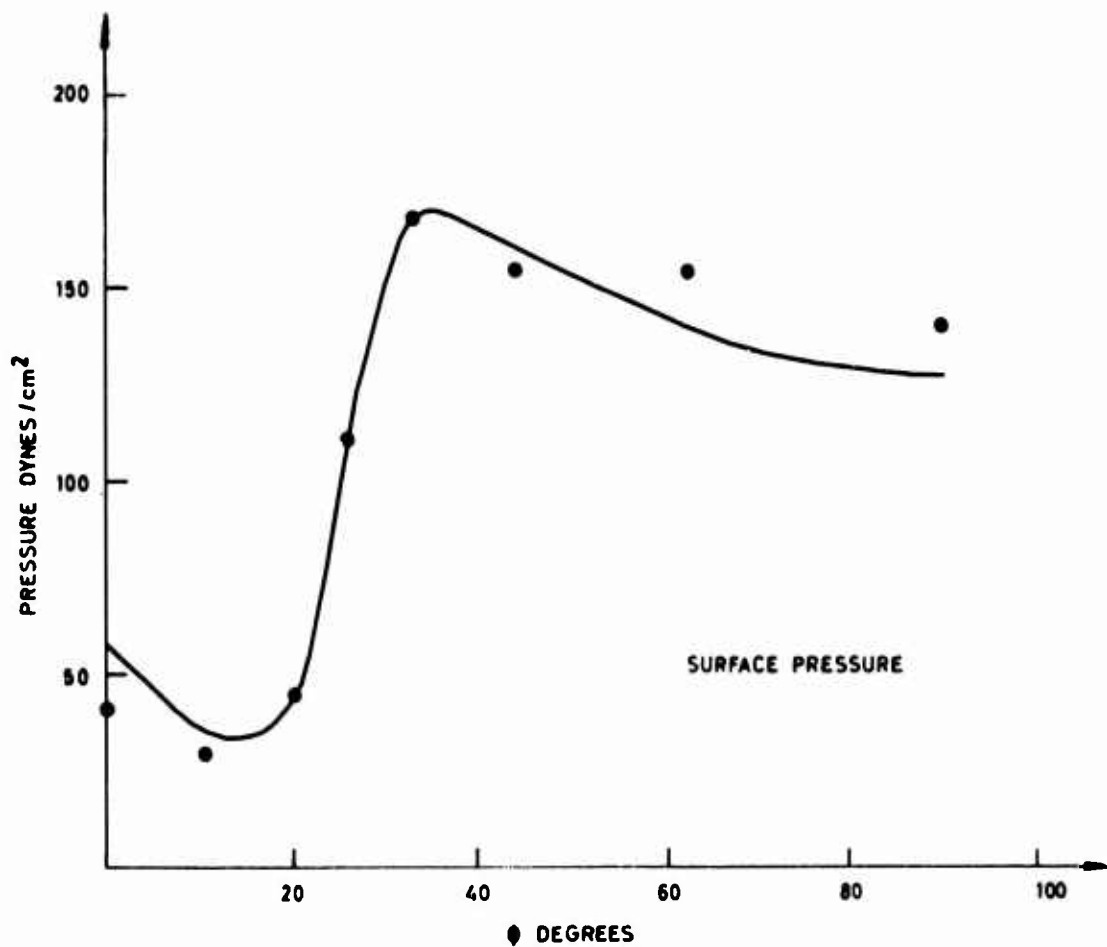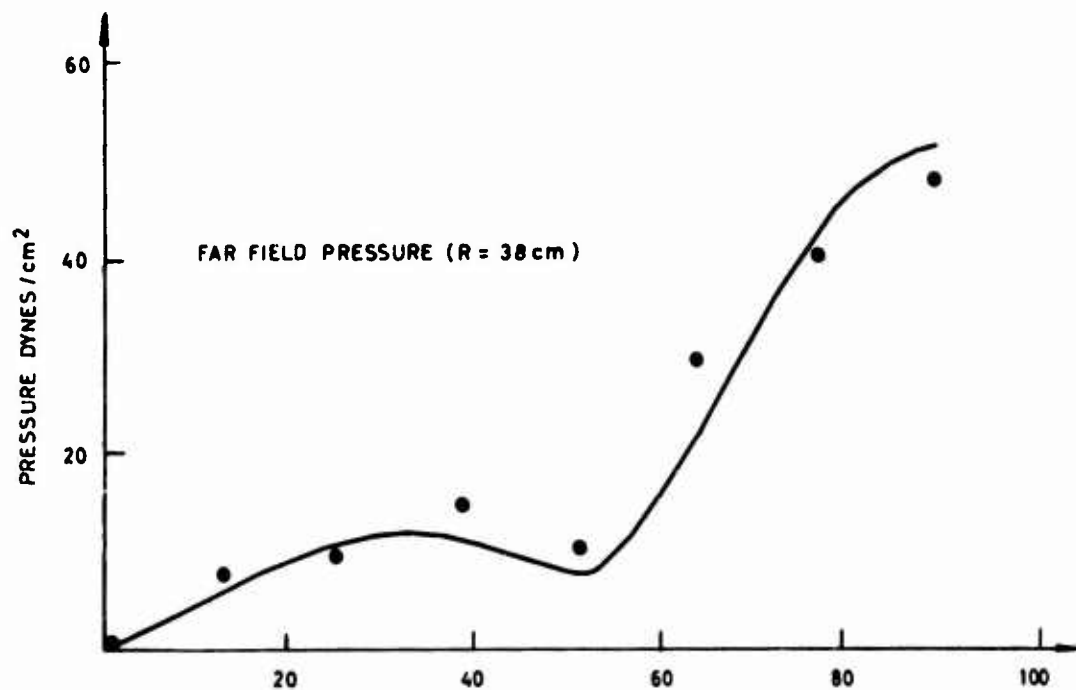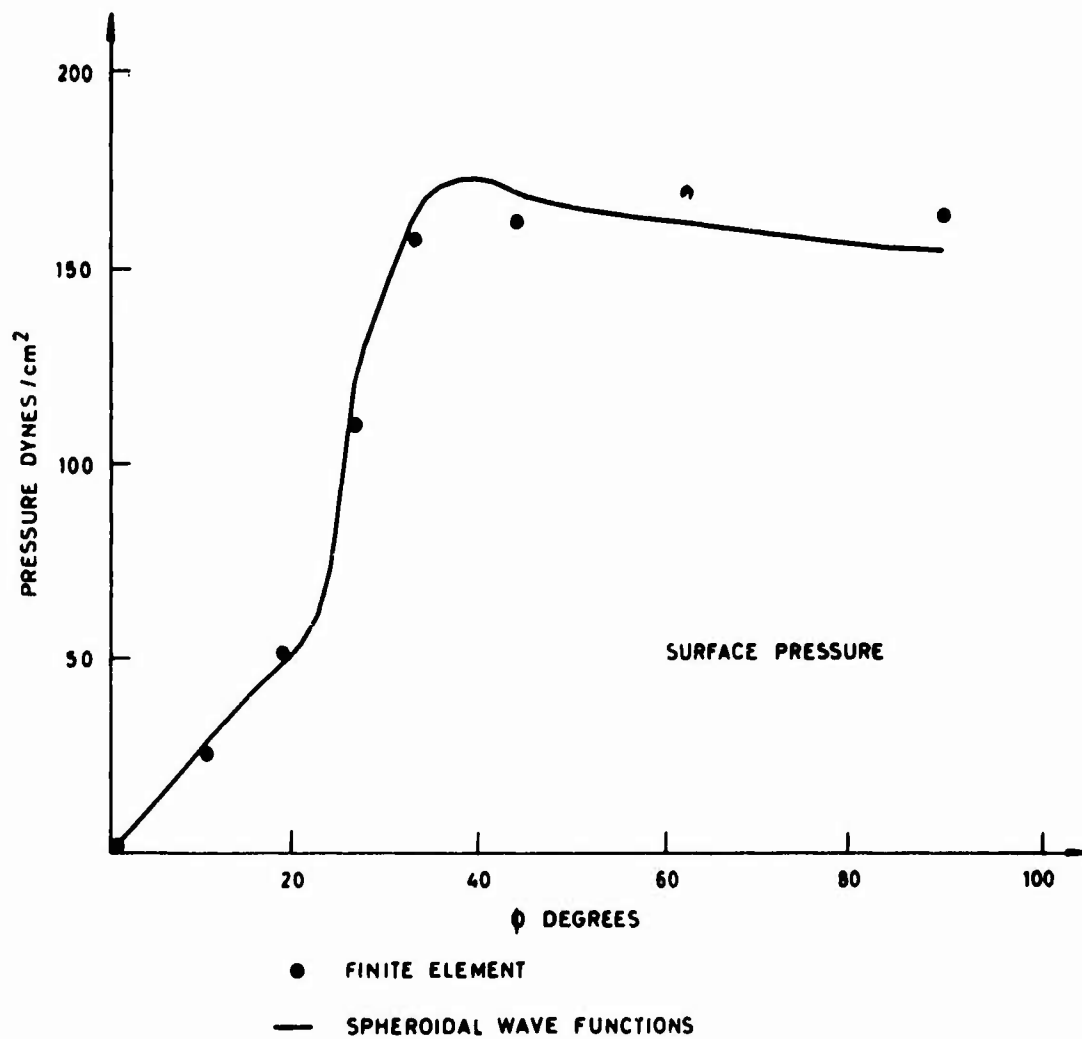FIG. 4   PRESSURE AMPLITUDE AT 10000Hz. N=O

FIG. 5   PRESSURE AMPLITUDE AT 10000 Hz, N=1